Discrete Opt Problem

$U = \{u_1, \dots, u_n\}$

$\forall S \subseteq U,$

$f(S) \to \mathbb{R}$

max / Min $f(S)$

S.t.

Constraints

e.g.

Vertex Cover

$U = \{v_1, \dots, v_n\}$

Select $S \subseteq U$

Min $|S|$

S.t.

$\forall (u, v) \in E,$

$u \in S$ OR $v \in S$

Monotonicity:

f is monotonic, if

$$f(S) \leq f(S \cup \{x\})$$

* Adding more elements to Set does not "hurt"

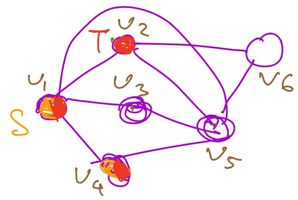
e.g.: adding more balls to a bowl doesn't reduce the number of Colors of ball

Submodularity

Let $S \subset T \subset U$, f is Submodular

if $\forall x \in U \setminus T$:

$$f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T)$$



Coverage$(\{v_1, v_4\}) = 5$

Coverage$(\{v_1, v_2, v_4\}) = 7$

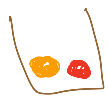
$$\text{Coverage}(S \cup \{v_6\}) = 7$$
$$\text{Coverage}(T \cup \{v_6\}) = 8$$

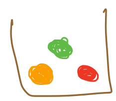
$$\text{Cov}(S \cup \{v_6\}) - \text{Cov}(S) = 7 - 5 = 2 = A$$
$$\text{Cov}(T \cup \{v_6\}) - \text{Cov}(T) = 9 - 8 = 1 = B$$

$$A \geq B \quad \checkmark$$

---

number of colors in a bowl

 $S \quad f(S) = 2$

 $T \quad f(T) = 3$

$$f(T \cup \{\bullet\}) - f(T) = 0$$
$$f(S \cup \{\bullet\}) - f(S) = 1$$

Theorem:

If the opt. function (f) is (i) monotonic and (ii) Submodular:

1) a greedy approach satisfies the approximation ratio of $(1 + 1/e)$

2) No other Alg. can do better than this unless $P = NP$

E.g. Max K_Cover

Max $\text{Coverage}(S)^*$

s.t. $|S| \leq k$

* The objective function is
1. monotonic
2. Submodular

$\Rightarrow$ Greedy Satisfies an approx-ratio of $\boxed{1/(1-1/e)}$

or: $\frac{App}{opt} \geq 1 - 1/e$

Set_Cover

Min $|S|$

s.t.

all elements are Covered

Greedy Satisfies approximation Ratio of $\boxed{\log(n)}$

** $\frac{1}{1-1/e} \simeq 1.58$ is a Constant approx. ratio, while $\log(n)$ is not Constant

Greedy Approximation Alg. for Max K_Cover

```
Sel = { }
Covered = { }
for i = 1 to k
    S_i = argmax{|S \ Covered|}
    add S_i to Sel
    Covered = Covered ∪ S_i
return Sel
```