Probablistic Method

1 – Any random variable with mean μ

- $\exists$ value $x \geq \mu$
- $\exists$ value $x \leq \mu$

2 – If an object with specific properties has a prob. higher than zero to be drawn from a sample pool, then the pool should contain at least One from that object.

e.g. if $P(\text{Red}) > 0 \Rightarrow \exists$ at least one red ball in the pool

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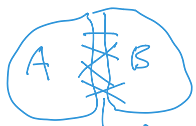
Max-Cut:



↳ Maximized

* for any graph, there exists a cut of at least $m/2$ edges.

→ $\forall$ vertex v_i put it in Cut A (or B) with prob. $1/2$. ← Algorithm.

$P(e_i \in \text{Cut Set}) = 1/2$

$$\Rightarrow E\big[|\text{Cut Set}|\big] = E\Big[\sum_{\forall e_i} P_i\Big] = \sum_{i=1}^{m} 1/2 = m/2 \checkmark$$

⇒ Based on prob. Method:

$\exists$ Cut s.t. $|\text{Cut Set}| \geq m/2$

Max-SAT:

+ give $C_1 \ldots C_m$ clauses and $v_1 \ldots v_n$ variables

every clause contains k literals

+ Obj: Find an assignment to $v_1 \ldots v_n$, s.t. # satisfied clauses is maximized.

$\boxed{\text{Randomized Rounding}}$

↳ Similar to LP-Relaxation

The Algorithms (LP-Relaxation for Max-SAT)

1. Formulate Max-SAT as IP

$$Z_i : \begin{cases} 1 & \text{clause } i \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

$$X_i : \begin{cases} 1 & \text{if } v_i \text{ is assigned to TRUE} \\ 0 & \text{" " " " FALSE} \end{cases}$$

$\forall$ clause C_i

C_i^+: the set of non-neg. literals in the clause

C_i^-: " " neg. "

$$\text{Max} \sum_{i=1}^{m} Z_i$$

s.t.

$\forall\ 1 \leq i \leq m$

$$\sum_{\forall v_j \in C_i^+} X_j + \sum_{\forall v_j \in C_i^-} (1 - X_j) \geq Z_i$$

$X_i \in \{0,1\}$

$Z_i \in \{0,1\}$

e.g. $C_i : V_1 \vee \bar{V}_3 \vee \bar{V}_4$ → $C_i^+ : \{V_1\}$, $C_i^- : \{V_3, V_4\}$

② Relax IP to LP, and Solve LP

Suppose
$\bar{z}_i$ is the value of z_i based on LP

$\bar{x}_i$ 〃 〃 〃 x_i

e.g.: $\bar{x}_i : 0.53$

③ for $i=1$ to n:

Set $x_i^* = 1$ with prob $\bar{x}_i$,

0 otherwise

$P(\text{Clause } C_i \text{ is Satisfied})$

$= 1 - P(C_i \text{ not Sat.})$

$= 1 - \prod (1 - \bar{x}_j)^*$

*: I assume all literals are non-neg.

C_i is empty

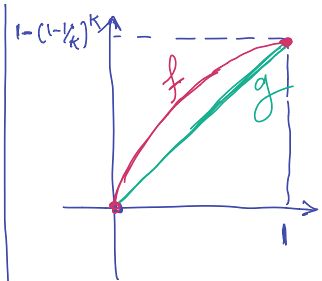
$\rightarrow C_i : \bar{x}_1 + \bar{x}_2 + \dots \bar{x}_k \geq \bar{z}_i$

$P(C_i \text{ is Sat.}) =$

$1 - \prod_{j=1}^{k} (1 - \bar{x}_j)$

$\geq 1 - \left(1 - \frac{\bar{z}_i}{k}\right)^k = f$

$g = \left(1 - (1 - 1/k)^k\right) \bar{z}_i$



$P(C_i \text{ Sat.}) \geq 1 - \left(1 - \frac{\bar{z}_i}{k}\right)^k$

$\geq \left(1 - (1 - 1/k)^k\right) \bar{z}_i$

$\boxed{\geq (1 - 1/e) \bar{z}_i}$ ✓

$E[|A|] = E\left(\sum_{i=1}^{m} P(C_i \text{ Sat.})\right)$

$\geq \sum_{i=1}^{m} (1 - 1/e) \bar{z}_i$

$= (1 - 1/e) \sum \bar{z}_i$

$* \sum_{i=1}^{m} \bar{z}_i \geq \underbrace{\sum z_i^\circ}_{\text{optimal assignment}}$

$E[|A|] \geq (1 - 1/e) \sum \bar{z}_i$

$\geq (1 - 1/e) \text{ Opt.}$ ✓

$\boxed{E[\text{Approx-Ratio}] = \frac{1}{1 - 1/e}}$

$E[|A|] \geq (1-1/e)\,Opt \geq 3/4\,Opt$

$\forall$ Max-Sat instance,

$\exists$ an assignment that satisfies at least $3/4$ of clauses

Using Prob. Method.