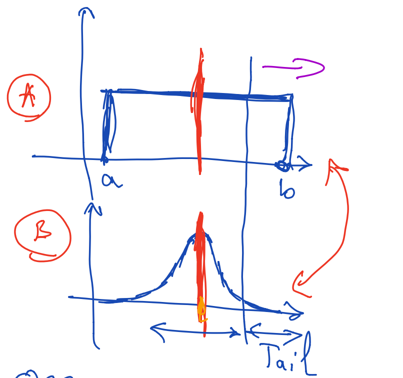

Occupancy Problem

assume

n balls

n bins

Equal Prob.

bin_1 bin_2 .... bin_n

$E[B_{i_n}] = 1$

→ # balls in bin_i after n balls

$P(b_j \to B_i) = 1/n$

what is the prob. that more than k balls fall into a bin?

$P(\mathcal{E}_i(k))$: Prob. that Bin_i has exactly k balls in it

$$P(\mathcal{E}_i(k)) = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^k$$

$$\leq \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\boxed{\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k} \quad (1)$$

using (1):

$$P(\mathcal{E}_i(k)) \leq \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\leq \left(\frac{ne}{k}\right)^k\left(\frac{1}{n}\right)^k$$

$$= \left(\frac{e}{k}\right)^k \checkmark$$

$P(\mathcal{E}^*_i(k))$: Prob. of having at least k balls in bin_i

$$P(\mathcal{E}^*_i(k)) \leq \sum_{j=k}^{n}\left(\frac{e}{j}\right)^j$$

$$\leq \sum_{j=k}^{n}\left(\frac{e}{k}\right)^j$$

$$= \left(\frac{e}{k}\right)^k\left(1 + \frac{e}{k} + \left(\frac{e}{k}\right)^2 + \dots\right)$$

$$\leq \left(\frac{e}{k}\right)^k\left(\frac{1}{1-e/k}\right) = A$$

if $k = O(\log n)$

$\to = \left\lceil \frac{3\ln n}{\ln(\ln(n))} \right\rceil$

$$\boxed{A \leq \frac{1}{n^2}} \checkmark$$

Union Bound

$$\boxed{P(\bigcup \varepsilon_i) \leqslant \sum P(\varepsilon_i)}$$

$$P\left(\bigcup \varepsilon_i^*(k)\right) \leqslant \sum_{i=1}^{n} P\left(\varepsilon_i(k)\right)$$

$$\leqslant \sum \left(e/k\right)^k \left(\frac{1}{1-e/k}\right)$$

if $k = \log n$

$$\leqslant \sum_{i=1}^{n} 1/n^2 = 1/n \checkmark$$

The chance of having a bin with more than $\underline{\log n}$ balls in it is less than $\boxed{1/n}$

with Prob. $(1-1/n)$ no bin has more than $\underline{k = \log n}$ balls in it.