PTAS: Polynomial Time Approx. Sol.
Given a problem X with input size n, and a value ε, $\varepsilon > 0$
provide a solution w/t approx-ratio of $(1+\epsilon)$ in a time polynomial in n

min $\frac{\text{approx}}{\text{opt}} \leq 1+\epsilon$

FPTAS: Fully PTAS

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Polynomial in n and $1/\varepsilon$

Subset Sum Problem

$U = \{u_1, u_2, \cdots, u_n\}$
target t

$S \subseteq U$ s.t $\sum_{\forall u_i \in S} u_i \leq t$

min $t - \sum_{\forall u_i \in S} u_i$

$$M[i,j] = \begin{cases} M[i-1,j] & \text{if } j < u_i \\ \min\{M[i-1,j], M[i-1,j-u_i]\} & \text{otherwise} \end{cases}$$

distance from optimum

$l_0 = \{\ \}$
for $i = 1$ to n
- $l_i = \underset{?}{\text{Merge}}(L_{i-1}, L_{i-1} \underset{?}{\oplus} u_i)$
- remove all values larger than t from l_i

return $l_{n,m}$ The last element

$\oplus$: $\{x_1, \cdots, x_m\} \oplus a$
$= \{a, x_1 + a, \cdots, x_m + a\}$

Merge: the merge function in Merge-Sort

e.g.
$[5, 3, 4, 2\}$ $t = 10$
$l_0 = \{\ \}$
$l_1 = \{5\}$
l_2: $l_1 \oplus 3 = \{3, 5+3\} = \{3, 8\}$
$l_2 = \{3, 5, 8\}$
$l_3 = \{3, 5, 8\} \cup \{4, 9, \cancel{12}\}$
$\{3, 4, 5, 8, 9\}$

$l_4 = \{3, 4, 5, 8, 9\} \cup \{2, 5, 6, 7, 10\}$
$= \{2, 3, \cdots, 9, \boxed{10}\}$
opt

The Approx. Algorithm

let $\delta = \varepsilon/2n$

$\boxed{\text{The idea is to let a value } y_j \text{ in a list } \ell_i \text{ represent all values in range } [y_j, y_j(1+\delta)]}$

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→ ℓ0 = { }
  for i=1 to n
   |- ℓi = Merge(ℓi-1, ℓi-1 ⊕ ui)
   |- remove all values larger than t
   |- Trim(ℓi, δ)
  return ℓn,m
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Trim(ℓ, δ)
    ℓ° = {ℓ[0]}, c = ℓ[0]
    for i=1 to |ℓ|
     | - if (ℓ[i] ≤ c(1+δ))
     |         continue
     | - c = ℓ[i]
     | - add ℓ[i] to ℓ°
    return ℓ°
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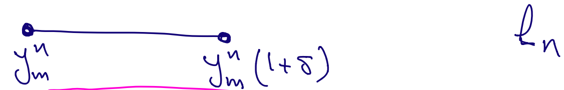
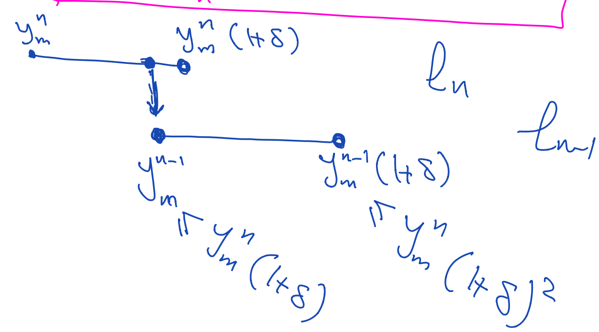
Theorem: The Approx algorithm satisfies the approx-ratio of $1+\varepsilon$

Proof:



$\boxed{y_m^n \text{ is the output of the Alg. It represents all values in range } [y_m^n, y_m^n(1+\delta)] \text{ in list } \ell_n}$



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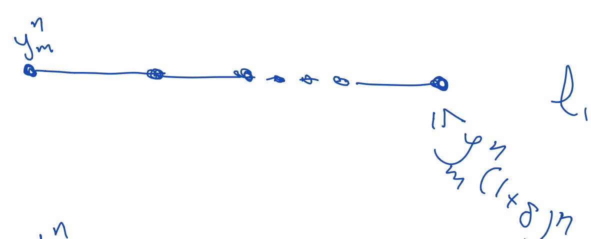

$y_m^n = \text{approx}$

$\text{approx} \leq \text{opt} \leq \text{approx}(1+\delta)^n$

$\text{opt} \leq \text{approx} \sum_{k=0}^{n} \binom{n}{k} \delta^k$

$= \text{approx}(1 + \varepsilon/2 + \cdots) \leq \text{approx}(1+\varepsilon)$

$\Rightarrow \frac{\text{opt}}{\text{approx}} \leq 1+\varepsilon$ ✓

Theorem: The Approx Alg. is polynomial in input size and $1/\varepsilon$

Proof:

$\boxed{m}$ → size of the list at a step i

$$y_1 (1+\delta)^{m} \leq t$$

$$\Rightarrow m \leq \log_{(1+\delta)} t/y_1 \leq \log_{(1+\delta)} t = \frac{\ln t}{\ln(1+\delta)} = A$$

Since $\delta > 0 \Rightarrow \ln(1+\delta) \geq \frac{\delta}{1+\delta}$

$$\Rightarrow A \leq \ln(t)\frac{1+\delta}{\delta} = \ln(t)\left(1+1/\delta\right) = \ln(t)\left(1+\frac{2n}{\varepsilon}\right)$$

- at every iteration the size of L_i is at most $\ln(t)(1+2n/\varepsilon)$
- The number of iterations is n

$\Rightarrow$ The running time is

$$n \ln(t)\left(1+\frac{2n}{\varepsilon}\right) = \boxed{O\left(n^2 \tfrac{1}{\varepsilon} \ln(t)\right)}$$