Probabilistic Method

1- Any random variable with mean μ

- $\exists$ value $x \geq \mu$
- $\exists$ value $x \leq \mu$

2- if an object/event with Specific Properties has a Prob. Larger than Zero to be drawn from a dist. there should exist at Least One instance of that object/event

e.g. in a bag of balls, if

$P(\text{red}) > 0$

$\hookrightarrow \exists$ a ball that is red.

Generalized Max SAT: Given m clauses $C_1, \dots, C_n$, each Consisting of K variables, Find the assignment that Max. the # of Satisfied Clauses.

Alg. 1: Randomly assign values to the variables:

$\Rightarrow$ Expected approx. ratio ≥ 2

$$P(\text{Satisfy } C_i) = 1 - 2^{-k}$$

$$E[\text{clauses Satisfied}] = \sum P(\text{Satisfying } C_i) = m(1-2^{-k})$$

$$\geq m/2$$

$$\Rightarrow \frac{\text{opt}}{\text{approx}} \leq 2$$

Alg. : Randomized Rounding (LP- Relaxation)

Step 1: Form IP

- $z_i = \begin{cases} 1 & \text{if } C_i \text{ is Satisfied} \\ 0 & \text{otherwise} \end{cases}$

- $x_i = \begin{cases} 1 & \text{if } v_i \text{ is Set to TRUE} \\ 0 & \text{" " " " " FALSE} \end{cases}$

$$\text{Max} \sum_{j=1}^{m} z_j$$

$$\text{s.t.} \quad \sum_{v_j \in C_i} x_j + \sum_{\bar{v}_j \in C_i} (1 - x_j) \geq z_i$$

$$x_i \in \{0,1\} \quad \forall\, 1 \leq i \leq n$$

$$z_i \in \{0,1\} \quad \forall\, 1 \leq i \leq m$$

Step 2: LP Relaxation:

Relax x_i to $x_i \in [0,1]$

" z_i to $z_i \in [0,1]$

$\{\langle x_1^* \cdots x_n^* \rangle, \langle z_1^* \cdots z_m^* \rangle\} \leftarrow$ Solve the LP

Step 3: Rounding (Randomized)

Goal is to assign values {True, False} to the variables

↳ Round $x_1^* \cdots x_m^* \longrightarrow x_1^+ \cdots x_n^+$

Binary

$\forall\, 1 \leq i \leq n$:

Set $x_i^+ = 1$ with Prob. x_i^*

" " " $= 0$ " " $(1 - x_i^*)$

Practical Suggestion:

Repeat Rounding Many times & Pick the assignment with max objective value

Many $\rightarrow O(n)$

objective value $\rightarrow \sum z_i^+$ (# clauses satisfied)

$E[\sum z_i^+]$: Expected # of Clauses that $x_1^+ \dots x_n^+$ Satisfy

$$P(x_i^+) = x_i^*$$

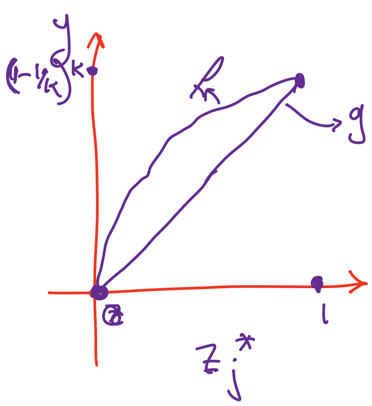
* Simplification: $C_j : x_{j1} \vee x_{j2} \vee \dots \vee x_{jk}$

$$P(C_j \text{ is Satisfied}) = 1 - P(C_j \text{ is not Satisfied})$$

$$= 1 - \prod_{v_i \in C_j} (1 - x_i^*)$$

$\sum x_i^* \geq z_i$

$\prod (1 - x_i^*)$ is Max when $x_i^* = z_i/k$



$$\geq \boxed{1 - \left(1 - \frac{z_j^*}{k}\right)^k} \rightarrow f(z_j^*)$$

$$\geq \boxed{\left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_j^*} \rightarrow g(z_j^*)$$

$$\left(1 + \frac{t}{k}\right)^k \leq e^t$$

$$P(C_j \text{ is Satisfied}) \geq (1 - 1/e)\, z_j^*$$

$$E[z_j^+] = \sum P(z_j^+) \geq \sum (1 - 1/e)\, z_j^*$$

$$= (1 - 1/e) \sum z_j^*$$

$$, \quad \text{Opt} \leq \sum z_j^*$$

$$\Rightarrow E[z_j^+] \geq (1 - 1/e)\, \text{OPT}$$

$$\Rightarrow \frac{E[z_j^+]}{\text{opt}} \geq (1 - 1/e)$$