Stable Marriage

Given n men and n women, each with a Pref. List of opposite sex, find a matching that is Stable. That is,

$$\nexists \begin{array}{ll} x - y & L(x): Y > y \\ & \text{AND} \\ X - Y, & L(Y): x > X \end{array}$$

1. Does a Stable Marriage always exist? YES

2. How to find a Stable Marriage?
   - Men Propose (Start from the top of their list)
   - Women Accept
     - At any moment that a woman recev.s a better Proposal they can change their selection
     - Men should stick to their Proposal as long as it is not rejected!

Runtime: $O(n^2)$ ⇐ worst-case (when all men make all Possible Proposals)

What is the AVERAGE Runtime of Stable Mar Alg.?

$O(n \log n)$

Search Space Size: $\underbrace{2n}_{\text{\# of Pref. lists}} \underbrace{(n!)}_{\text{Possible Permutations}}$

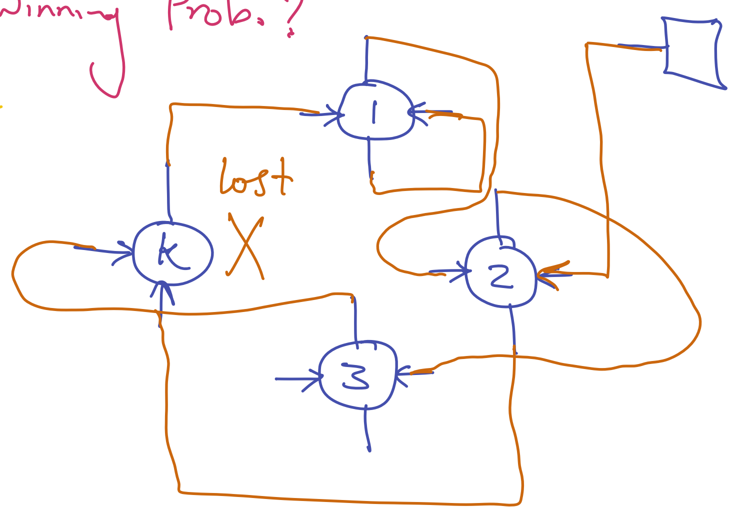
Principle of Deffered Decisions

Consider a card Game, (52 cards), Shuffled, Partitioned into 13 piles of 4 Cards

Draw a card from the (King Pile) and follow Picking from Piles specified by selected Cards. You win if you can Collect all 52 Cards.

what is the winning Prob.?

Simplified 1, 2, 3, k



Observation: The game is won iff the last card is a KING!

$\Rightarrow P(\text{win}) = 1/\#\text{Card-Types}$

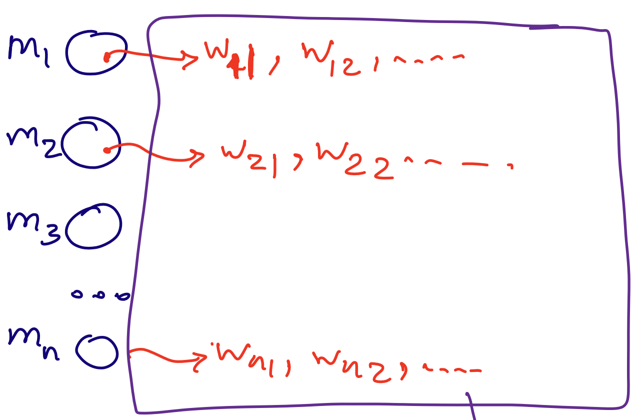
Observation: The Algorithm Stops when every woman Receives a proposal.

$\Rightarrow$ Expected Cost: exp. # of Proposals to make such that the name of all women are generated

$\Rightarrow$ Simplifiation:

Men Select a random woman at every iteration indep. of prev. choices.

$$C_{\text{simplified}} \geq C_{\text{Stable Marriage}}$$



how many Proposals to make in order to generate all names

Coupon Collector's Problem

– Given n Coupon-Types, n generators, Every time they generate a Coupon $\overbrace{}^{\text{Type}}$ uniformly at random,

what is the expected number of Coupons one need to collect, before all Coupon-Types are collected.

epoc i : The set of Coupons collected before the i-th fresh Coupon is discovered

epoc $\emptyset$:

$$P(\text{Success}_0) = 1 \qquad E[T_0] = 1$$

# of queries in epoc$_i$

epoc 1 :

$$P(\text{Success}_1) = \frac{n-1}{n} \qquad E[T_1] = \frac{n}{n-1}$$

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epoc i :

$$P(\text{Success}_i) = \frac{n-i}{n} \qquad E[T_i] = \frac{n}{n-i}$$

epoc $n-1$:

$$P(S_{n-1}) = \frac{1}{n} \qquad E[T_{n-1}] = n$$

$$E[T] = \sum_{i=0}^{n-1} E[T_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=0}^{n-1} \frac{1}{n-i}$$

$$= n\left(1/n + \frac{1}{n-1} + \cdots + 1/1 \right) = n \sum_{i=1}^{n} 1/i = n H_n$$

$$= O(n \log n)$$

$$P(T \geq \alpha\, n \log n) \leq \frac{n \log n}{\alpha\, n \log n} = 1/\alpha$$

Using Markov inequality

Variance

$$p_i = \frac{n-i}{n} \qquad \sigma_i^2 = \frac{1-p}{p^2} = \frac{i/n}{\frac{(n-i)^2}{n^2}} = \frac{in}{(n-i)^2}$$

$$\sigma^2 = \sigma^2\left(\sum T_i\right) = \sum_{i=0}^{n-1} \sigma_i^2$$

$$= \sum \frac{in}{(n-i)^2} = n \sum_{i=0}^{n-1} \frac{i}{(n-i)^2}$$

$$= n\left(\frac{1}{(n-1)^2} + \frac{2}{(n-2)^2} + \cdots + \frac{n-1}{(1)^2}\right)$$

$$= n \sum_{i=1}^{n-1} \frac{n-i}{i^2} = n\left(\sum n/i^2 - \sum 1/i\right)$$

$$= n^2 \sum_{i=1}^{n-1} 1/i^2 - n H_n = O(n^2)$$

$$\boxed{\lim_{n \to \infty} \sum_{i=1}^{n} 1/i^2 = \frac{\pi^2}{6}}$$

$\Rightarrow$ Using Chebyshev Ineq.

$$P(|T - n\log n| > xn) \leq 1/x^2$$