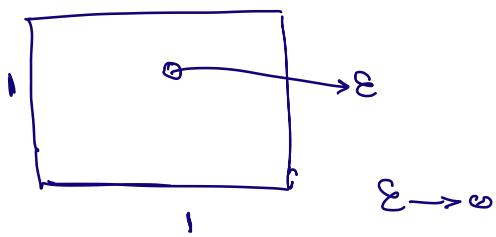


$P(\text{Success}) = \varepsilon$

- $E[\text{Success}] = \varepsilon$

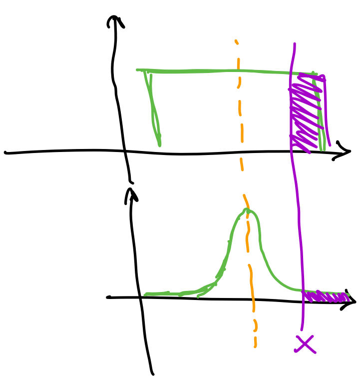
Repeat 100 times

$E[\#\text{Samples in Circle}] = 100\varepsilon$

- ∅ falls inside (high Prob)
  Estimated Area = ∅ (underestimating)
- at least 1 falls inside
  Estimated Area $\geq 0.01 \gg \varepsilon$
  (overestimate)

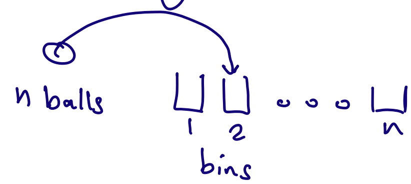
we want to design alg. that with a high Prob. their Runtime is less than a given value

Tail Prob.



$P(T > x) \leq \varepsilon$

Occupancy Problems



- given n balls, throw the balls in the bins independendly.

$\Rightarrow$ Expected # balls in each bin $= 1$

Claim: The Prob. of having more than $\log(n)$ balls in any bin is very low.

Proof:

$P(\text{having exactly } k \text{ balls in bin } i) = P(B_i = k)$

$$P(B_i = k) = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k}$$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

$$\Rightarrow P(B_i = k) \leq \left(\frac{ne}{k}\right)^k \left(\frac{1}{n}\right)^k = \left(\frac{e}{k}\right)^k$$

Using Union Bound

$$P(B_i \geq k) \leq \sum_{i=k}^{n} \left(\frac{e}{i}\right)^i$$

Union Bound

$$P\left(\bigcup_{i=1}^{n} \mathcal{E}_i\right) = P(\mathcal{E}_1) + P(\mathcal{E}_2) + \cdots + P(\mathcal{E}_n)$$
$$- P(\mathcal{E}_1 \cap \mathcal{E}_2) - P(\mathcal{E}_1 \cap \mathcal{E}_3) - \cdots - P(\mathcal{E}_{n-1} \cap \mathcal{E}_n)$$
$$+ P(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) \cdots$$

$$P\left(\bigcup_{i=1}^{n} \mathcal{E}_i\right) \leq \sum_{i=1}^{n} P(\mathcal{E}_i)$$ ← Union Bound.

$$P(B_i \geq k) \leq \sum_{i=k}^{n} \left(\frac{e}{i}\right)^i \leq \sum_{i=k}^{n} \left(\frac{e}{k}\right)^i$$

$$= \left(\frac{e}{k}\right)^k + \left(\frac{e}{k}\right)^{k+1} + \cdots + \left(\frac{e}{k}\right)^n$$

$$\leq \left(\frac{e}{k}\right)^k \left(1 + \frac{e}{k} + \left(\frac{e}{k}\right)^2 + \cdots\right)$$

$$\leq \left(\frac{e}{k}\right)^k \frac{1}{1 - e/k}$$

$$\boxed{k > e}$$

Using WolframAlpha,

if $k = O(\log n)$
$= \lceil (3 \ln n) / \ln \ln n \rceil$

$$P(B_i \geq k) \leq \left(\frac{e}{k}\right)^k \frac{1}{1 - e/k}$$
$$\leq 1/n^2$$

$$P(B_i^* \geq k) \leq ?$$

$$B_i^* = \max_{i=1}^{n} B_i$$

$$P(B_i^* \geq k) = P\left(\bigcup B_i \geq k\right)$$
$$\leq \sum_{i=1}^{n} P(B_i \geq k)$$
$$= n \times 1/n^2 = 1/n$$

$\Rightarrow$ P(having a bin with more than log(n) balls) $\leq 1/n$