R-QSort($x_1 \cdots x_n$)

$i$ = Uniform[1, n]

Pivot = $x_i$

$j$ = Partition($x_1 \cdots x_n$, Pivot)

R-QSort($x_1 \cdots x_{j-1}$)

R-QSort($x_{j+1} \cdots x_n$)

Expected Runtime of R-QSort is in $O(n \log n)$.

Observation 1:

$\forall i, j \leqslant n$, $x_i, x_j$ get Compared at most once.

Observation 2:

Runtime = The # of Pairs that get compared.

$\alpha_{ij}$ ≐ random variable that is 1 if $x_i$ and $x_j$ get Compared

$$\text{Runtime} = \sum_{\substack{\forall i,j \leqslant n \\ i \neq j}} \alpha_{ij}$$

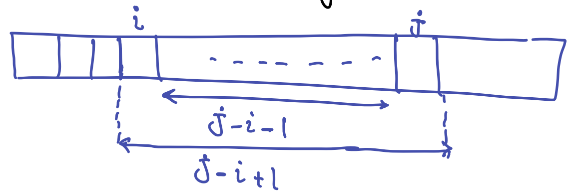
$$\alpha_{ij} = \begin{cases} 1 & P_{ij} \\ 0 & \end{cases}$$

$$E[x_i, x_j \text{ get Compared}] = 1 \times P_{ij} + 0 \times (1 - P_{ij}) = P_{ij}$$

Expected Runtime

$$E\left[\sum \alpha_{ij}\right] = \sum E[\alpha_{ij}]$$

$$= \sum P_{ij}$$

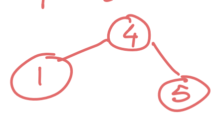



e.g.

| 1 | 4 | 5 |

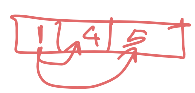
$\alpha_{1,3}$

$x_2$ is pivot




$\Rightarrow \alpha_{1,3} = \emptyset$

$x_1$ is Pivot




$\alpha_{1,3} = 1$

$x_3$ is Pivot


$\alpha_{1,3} = 1$

$\alpha_{i,j}$ is Zero only if any of $x_{i+1} \cdots x_{j-1}$ is selected as Pivot before $x_i$ or $x_j$

$\Rightarrow P_{ij} = P(x_i \text{ or } x_j \text{ is Selected as Pivot, befor } x_{i+1} \cdots x_{j-1})$

$$= \frac{2}{j - i + 1}$$

$$T_n = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_{ij}$$

$$E[T_n] = E\left[\sum\sum \alpha_{ij}\right]$$

$$= \sum\sum E[\alpha_{ij}] = \sum\sum p_{ij}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = 2\sum\sum \frac{1}{j-i+1}$$

$$= 2 \sum_{i=1}^{n-1} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-i+1} \right)$$

$$= 2 \sum_{i=1}^{n-1} \sum_{j=2}^{n-i+1} \frac{1}{j}$$

$$= 2 \sum_{i=1}^{n-1} H_{n-i+1}$$

$$\leq 2 \sum_{i=1}^{n-1} H_n$$

$$= O(n H_n)$$

$$= O(n \log n)$$