Discrete Optimization Problem

$U = \{u_1, u_2, \dots, u_n\}$

$\forall\, S \subseteq U$

$f : S \to \mathbb{R}$

objective:

$\max f(S)$

s.t.

Constraints

e.g.

max - vertex cover

$V = \{v_1 \dots v_n\}$

$\forall\, S \subseteq V$

$Cov(S) = \#$ edges S hit

$\max Cov(S)$

s.t.

$|S| \leq K$

Monotonicity:

f is monotonic, if

$$f(S \cup \{x\}) \geq f(S)$$

e.g.

max - Vertex - Cover

$Cov(S)$ is monotonic

Submodularity:

f is Submodular, if adding an element to a subset of S has higher benefit than adding it to S :

$$\forall S \subseteq U,\ \forall T \subseteq S:\ f(S \cup \{x\}) - f(S) \leq f(T \cup \{x\}) - f(T)$$

e.g. 1: U: ~~A~~ a set of Balls, each w/t a Color

$f(S)$: The # Colors in S

$S = \{ \quad \}$

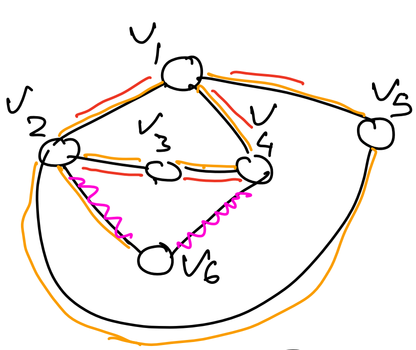
$T = \{ \quad \}$

$$f(S \cup \{x\}) - f(S) \leq f(T \cup \{x\}) - f(T)$$

e.g. 2, max-Vertex-Cover:

is Cov Submodular?

Yes;



$S = \{v_1, v_2, v_3\}$

$f(S) = 7$

$T = \{v_1, v_3\}$

$f(T) = 5$

$$f(S \cup \{v_6\}) - f(S) = 1$$

$$f(T \cup \{v_6\}) - f(T) = 2$$

Theorem: if the objective function f is
(1) monotonic, and (2) Submodular, then
(a) The greedy approach satisfies an $(1 - 1/e)$ approx. ratio
and (b) no other P-alg. can do better, unless $P = NP$.

$\boxed{\begin{array}{l} * \ (1-1/e) \text{ is a deviation of approx. ratio definition.} \\ \dfrac{1}{1-1/e} \text{ is the standard form} \end{array}}$

Max-Cover Problem

$$\text{Max } \Big| \bigcup_{S_i \in \mathcal{S}} S_i \Big| = f(x)$$

Coverage

S.t.

$$|\mathcal{S}| \leq k$$

Set-Cover Problem

$$\text{Min } |\mathcal{S}|$$

S.t.

$$\Big| \bigcup_{S_i \in \mathcal{S}} S_i \Big| = |U|$$

Approx. ratio for Greedy is $O(\log n)$

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(1) $f(x)$ is monotonic: adding a new set to $\mathcal{S}$ will not reduce $|\bigcup S_i|$

(2) $f(x)$ is Submodular:

$\forall T \subseteq \mathcal{S} \subseteq U$

$$f(T \cup \{x\}) - f(T) \geq f(\mathcal{S} \cup \{x\}) - f(\mathcal{S})$$

$$\Big| \bigcup_{S_i \in T \cup \{x\}} S_i \Big| - \Big| \bigcup^{T} \Big| \geq \Big| \bigcup^{\mathcal{S} \cup \{x\}} \Big| - \Big| \bigcup^{\mathcal{S}} \Big|$$

✓ Case 1: $x \in \bigcup^{T} \Rightarrow \Big| \bigcup^{T \cup \{x\}} \Big| = \Big| \bigcup^{T} \Big| \Rightarrow \Big| \bigcup^{T \cup \{x\}} \Big| - \Big| \bigcup^{T} \Big| = \emptyset$

$$\Big| \bigcup^{\mathcal{S} \cup \{x\}} \Big| = \Big| \bigcup^{\mathcal{S}} \Big| \Rightarrow \Big| \bigcup^{\mathcal{S} \cup \{x\}} \Big| - \Big| \bigcup^{\mathcal{S}} \Big| = \emptyset$$

Case 2: $x \notin \bigcup^{T}$, $x \in \bigcup^{S}$ $\Rightarrow$ $\left|\bigcup^{T \cup \{x\}}\right| - \left|\bigcup^{T}\right| = 1$

$$\left|\bigcup^{S \cup \{x\}}\right| - \left|\bigcup^{S}\right| = \emptyset$$

Case 3: $x \notin \bigcup^{T}$, $x \notin \bigcup^{S}$ $\Rightarrow$ $\left|\bigcup^{T \cup \{x\}}\right| - \left|\bigcup^{T}\right| = 1$

$$\left|\bigcup^{S \cup \{x\}}\right| - \left|\bigcup^{S}\right| = 1$$

$\Longrightarrow$ Greedy Satifies $(1 - 1/e)$-approx. ratio for Max-Cover Problem.