# A FPTAS for Subset Sum.

$U = \{I_1, \dots, I_n\}$, a target value $t$

Objective:

find $S \subseteq U$ s.t. $\sum_{I_j \in S} I_j = t$

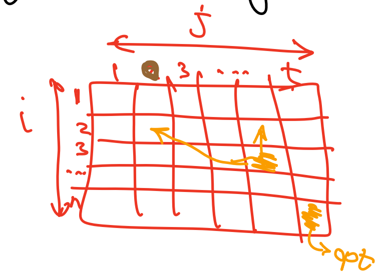
Optimization Version

find $S \subseteq U$ where

$\sum_{I_j \in S} I_j - t$ is minimized

$M[i, j]$: The optimal solution when considering the first $i$ numbers in $U$, and the target value is $j$

$\hookrightarrow \sum I_i \leqslant t$




$opt = M[n, t] - t$

$$M[i,j] = \begin{cases} M[i-1, j] & \text{if } j < I_i \\ \text{Max}\{\underbrace{M[i-1, j - I_i]}_{\text{Select } I_i}, \underbrace{M[i-1, j]}_{\text{Skip } I_i}\} \end{cases}$$

$L_0 = \{\}$

for $i = 1$ to $n$

- $L_i = \text{Merge}(L_{i-1}, L_{i-1} \oplus^{*} I_i\}$
- remove all values larger than $\underline{\underline{t}}$ from $L_i$

retur $L_{n,max}$

$\oplus(L_{i-1}, I_i)$:

for $j = 1$ to $|L_{i-1}|$:

$L_{i-1}[j] = L_{i-1}[j] + I_i$

e.g.

$U = \{5, 4, 3, 2\}$

$t = 10$

$L_0 = \{0\}$

$L_1 = \{0, 5\}$

$L_2 = \text{Merge}(L_1, L_1 \oplus 4)$

$L_1 \oplus 4 = \{0+4, 5+4\} = \{4, 9\}$

$L_2 = \{0, 4, 5, 9\}$

$L_3 = \{\{0, 4, 5, 9\}, \{3, 7, 8, \cancel{12}\}\}$

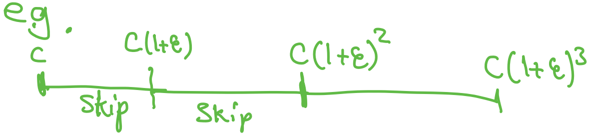
$= \{0, 3, 4, 5, 7, 8, 9\}$

$L_4 = \{L_3, L_3 \oplus 2\} = \{0, \dots, 10\}$

Select $\{I_1, I_3, I_4\}$

Exponential

e.g.




$L_0 = \{0\}$ ; $\delta = \varepsilon / 2n$

for $i = 1$ to $n$

- $L_i = \text{Merge}(L_{i-1}, L_{i-1} \oplus I_i)$
- remove all numbers larger than $t$
- $\text{Trim}(L_i, \delta)$

return $L_{n,max}$

$\text{Trim}(L, \delta)$

$c = L[1]$ ; $i = 2$

while $i \leq n$

  while $L[i] \leq c(1+\delta)$

    delete $L[i]$ from $L$

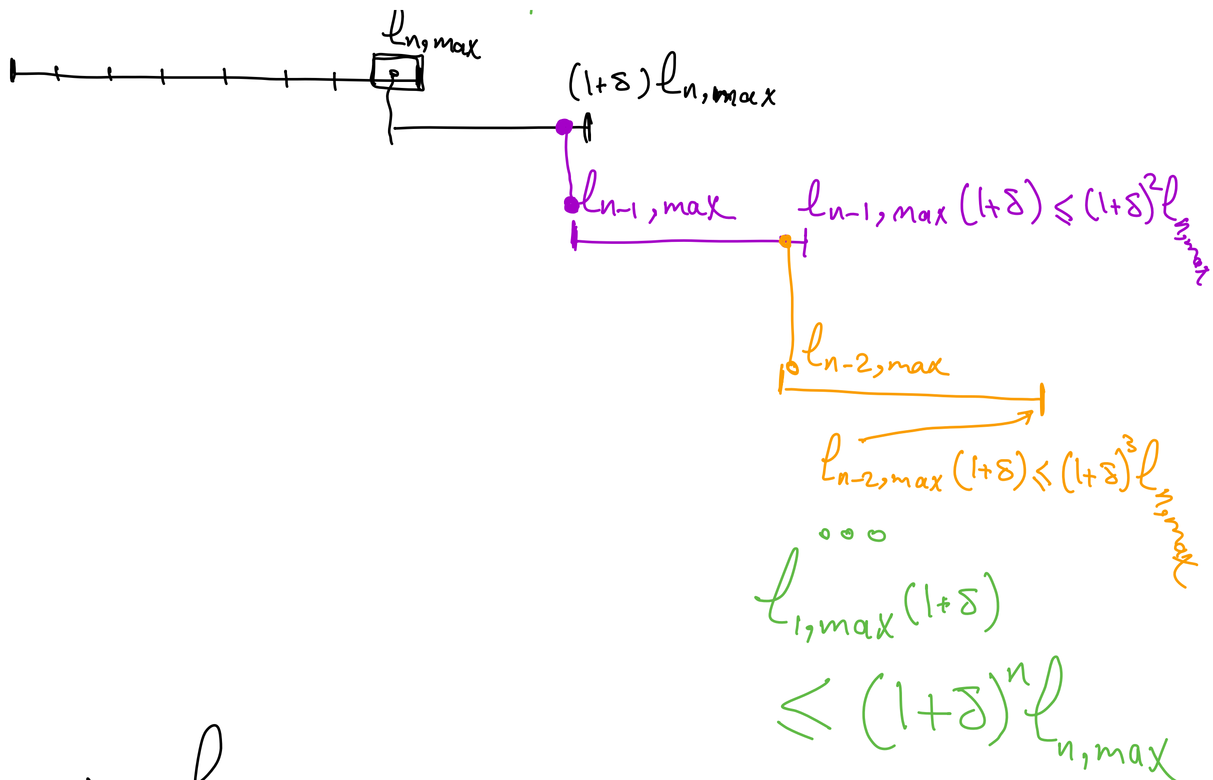
  $c = L[i]$

  $i{+}{+}$

return $L$

$L_3 = \{0, 3, 4, 5, 7, 8, 9\}$  $1+\delta = 1.5$

$\{0, 3, \cancel{4}, 5, \cancel{7}, 8, \cancel{9}\}$




$$\text{Approx} = \ell_{n,max}$$

$$OPT \leq (1+\delta)^n \,\text{Approx} \Rightarrow \frac{OPT}{\text{Approx}} \leq (1+\delta)^n$$

$$(1+\delta)^n = \sum_{k=0}^{n} \binom{n}{k} \delta^k$$

$$= 1 + n\delta + \frac{n^2}{2}\delta^2 + \cdots$$

$$= 1 + \cancel{n}\frac{\varepsilon}{2\cancel{n}} + \frac{\cancel{n^2}}{2}\frac{\varepsilon^2}{2^2\cancel{n^2}} + \cdots$$

$$= 1 + \varepsilon/2 + \frac{\varepsilon^2}{2^3} + \cdots \leq (1+\varepsilon)$$

Time Complexity

Iteration $i$

$$\ell_{i,1} \to \ell_{i,2} \to \ell_{i,3} \to \ell_{i,4} \to \cdots \to \ell_{i,m}$$

$\ell_{i,2} \geq \ell_{i,1}(1+\delta)$

$\ell_{i,3} \geq \ell_{i,2}(1+\delta) \geq \ell_{i,1}(1+\delta)^2$

$\ell_{i,4} \geq \ell_{i,1}(1+\delta)^3$

$\ell_{i,m} \geq (1+\delta)^{m-1}$

① $\ell_{i,m} \geq (1+\delta)^{m-1} \ell_{i,1}$

② $\ell_{i,m} \leq t$

①,② $\Rightarrow \ell_{i,1}(1+\delta)^{m-1} \leq t$

$$\Rightarrow m-1 \log(1+\delta) \leq \log t/\ell_{i,1} \leq \log t$$

$$m-1 \leq \frac{\log t}{\log(1+\delta)}$$

$\delta > 0$

$$\Rightarrow \log(1+\delta) \geq \frac{\delta}{1+\delta}$$

$$m-1 \leq \log t \left(\frac{1+\delta}{\delta}\right) = \log t \left(1 + 2n/\varepsilon\right)$$

for $n$ iterations

$$n \log t \left(1 + 2n/\varepsilon\right) = O\left(n^2 \log t \ 1/\varepsilon\right)$$