LP-Relaxation

IP	LP
Min $f(x)$	Min $f(x)$ s.t.
$\underset{m\times n}{A} x \leq b$	$Ax \leq b$
$\forall x_i \in X$	$\forall x_i \in X$
$x_i \in \mathbb{Z}$	$x_i \in \mathbb{R}$
NP-Complete	P
$f(x^{\perp})$ $\geq$	$f(x^*)$

$\boxed{\text{Similarly, for Max, LP provides an upper bound on the opt. Solution of IP}}$

LP Relaxation

Step 1: Model your Problem as IP

Step 2: Relax IP to LP

Step 3: $X^* \leftarrow \text{Solve}(LP)$

Step 4: "Round" X^* to integer numbers X^+

VC-approximation Using LP-relaxation

V-C

$v_1, \dots v_n$: vertices

$e_1, \dots e_m$: edges

Let $x_i \in \{0,1\}$ be 1 if v_i is selected, 0 otherwise

Step 1

$$\begin{aligned} &\text{Min} \quad \sum x_i \\ &\text{S.t.} \\ &\quad \forall (v_i, v_j) \in E \\ &\qquad x_i + x_j \geq 1 \\ &\quad \forall i \in [1, n] \\ &\qquad x_i \in \{0,1\} \end{aligned}$$

Step 2 Relax to LP

$$\begin{aligned} &\text{Min} \quad \sum x_i \\ &\text{S.t.} \\ &\quad \forall (v_i, v_j) \in E \\ &\qquad x_i + x_j \geq 1 \\ &\quad \forall i \\ &\qquad x_i \in [0, 1] \end{aligned}$$

Step 3 Solve (LP)

X^* is the optimal values for LP

Step 4 : Rounding

$X_i^+ = \text{round}(X_i^*)$ to closest integer

approx. Solution

Correctness

$\forall (v_i, v_j) \quad x_i^* + x_j^* \geq 1 \Rightarrow$ at least one of x_i^* or x_j^* should be Larger (or equal) to 0.5

$\Rightarrow$ at least one of x_i^+ or x_j^+ should be 1

LP-relaxation provides a 2-approx. alg. for v-c.

$$OPT \geq \sum x_i^* \quad \text{(The opt. solution for LP)}$$

$$= \sum_{x_i^* < 0.5} x_i^* + \sum_{x_j^* \geq 0.5} x_j^*$$

$$\geq \sum_{x_j^* \geq 0.5} x_j^*$$

$$\geq \sum_{x_j^* \geq 0.5} 0.5 = 0.5 \sum_{x_j^* \geq 0.5} 1 = 0.5 \|A\|$$

$$\Rightarrow OPT \geq 0.5 \|A\|$$

$$\Rightarrow \frac{\|A\|}{Opt} \leq 2$$

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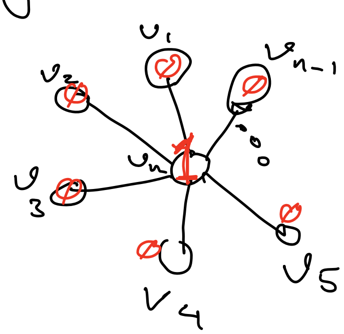
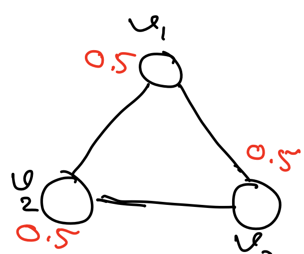
e.g.

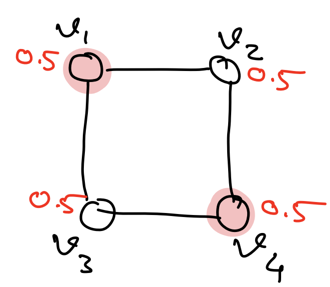




approx: $\{v_1, v_2, v_3\}$

Opt: $\{v_1, v_2\}$

$3/2$



approx: $\{v_1, v_2, v_3, v_4\}$

opt: $\{v_1, v_4\}$

$4/2 = 2$

approx. ratio is Tight

Weighted Vertex Cover :

- every vertex is associated wit a weight
- Goal: min $\sum w_i x_i$

LP-relaxation:

Change the obj. function to

$$\min \sum w_i x_i$$