Dynamic Necklace Splitting

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What is necklace splitting?

We want to cut the necklace such that the resulting intervals can be fairly distributed between the agents: each agent should receive the same number of red beads and the same number of blue beads.

Applications

- Fair hash maps: partition data (beads) across buckets (agents) such that each bucket receives an equal share of each attribute type (Shahbazi et al., SIGMOD '24)
- Load-balancing: spread tasks (beads) across servers (agents) where tasks need to be completed in a specified order
- Bucketization for fairness in ML: training and test data should have similar minority representation

Static necklace splitting

- m beads (m_1 red, m_2 blue)
- k agents
- Goal: efficiently cut the necklace in as few places as possible such that the resulting intervals can be distributed such that each agent receives m_1/k red beads and m_2/k blue beads
- Optimal solution: 2(k-1) cuts (Alon and Graur, ICALP '21)

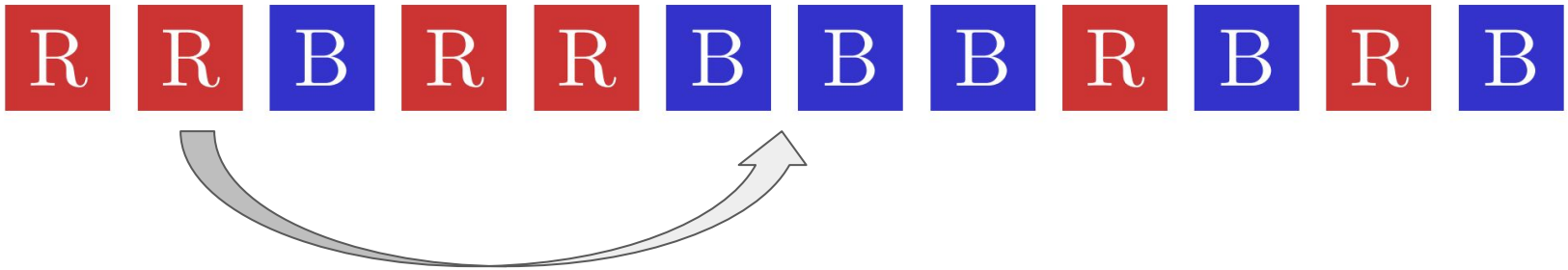
Static necklace splitting (cont.)

Offline algorithm (Shahbazi et al., SIGMOD '24):

- Consider range of length m/k
- Move this range bead by bead until it contains exactly m_1/k red beads
- Allocate the range to the current agent
- Consider the remaining beads and the next agent — repeat

Dynamic setting

- Relocation: move a bead to an arbitrary position in the necklace
- Insertion: insert *k* beads of the same color at arbitrary positions in the necklace
- Deletion: remove *k* beads of the same color from the necklace



Key results

Exact algorithm:

- 2(k-1) cuts
- Running time: linear in m and k

Approximate algorithm:

- Approximately fair w.h.p.
- 2(k-1) cuts
- Running time: polylogarithmic in m for fixed k

Exact algorithm: relocation

We will start by designing an algorithm for relocation, and later we will extend it to handle insertion/deletion.

Exact algorithm: adjacent positions

If we need to relocate a bead to an adjacent position:

- Remove all cuts between the agents corresponding to the starting and ending positions of the bead
- Rerun the offline algorithm on those two agents

Exact algorithm: nonadjacent positions

If we need to relocate a bead to a nonadjacent position:

- Model the set of agents as a graph with edges between agents with shared boundaries
- Find the shortest path between the agents corresponding to the starting and ending positions of the bead
- Remove all cuts between the agents on the path
- Rerun the offline algorithm on those agents

Exact algorithm: batch updates

Finally, we can perform batch updates to increase efficiency:

- Construct a flow network:
  - Agents to give beads are sources
  - Agents to receive beads are sinks
- Find a max flow with the number of active nodes minimized
  - This problem is NP-hard
  - Efficient 2-approximate solution possible on a spanning tree
- Remove all cuts between active agents
- Rerun the offline algorithm on those agents

Running time: linear in m and k

Exact algorithm: insertion/deletion

Now that we have an algorithm for relocation, we can use it to solve insertion/deletion.

- Insertion: give each agent a new bead, and then relocate the beads to the desired positions
- Deletion: remove a bead from each agent, and then relocate the beads to be deleted to the empty spots

Approximate algorithm

- Construct a binary search tree over the necklace
- To perform a dynamic update, we relocate/insert/delete the relevant beads in the tree in O(log m) time
- To generate the set of cuts on demand:
  - Sample subsets of beads and run the offline algorithm on those subsets
  - This gives us an approximately fair solution in polylogarithmic time (for fixed k)

Takeaways

- Applications: fair hash maps, load-balancing, and bucketization
- Exact solution with optimal 2(*k*-1) cuts generated in linear time
- Approximately fair solution generated in polylogarithmic time for fixed *k*

link to paper

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